Properties of Commutator Submultigroups

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Abstract

This paper vividly studies the properties of commutator in multigroups. It was shown that a commutator submultigroup is commutative and normal. Further, we show that the commutator of the homomorphic image equals the image of the commutator and the result also holds for the inverse image..

Keywords: *Multigroup, Submultigroup,Commutator submultigroups.*

1. Introduction

Modern mathematics has been expanded by violating the basic rules that guide cantorial set which is the foundation for algebra. Examples of such set are; fuzzy, soft, rough, intuitionistic, multiset etc. Similarly, the theory of multisets in ([2], [3], [6]) has been studied by supposing that for a considered set, an element can occur many times which violate the idea of distinct collection of objects.

The concept of multigroups via multiset was introduced in [12] and some related properties were established. Since then, several authors have explored the concept of multigroups (see for details [1], [4], [5], [7], [8], [9], [13], [14], [15] and [16]). Due to the concept of multisets, a submultigroup can be regular, irregular, complete and incomplete. The notion of commutator of two submultigroups was introduced in [10] and proved that the commutator of two normal submultigroups of a multigroup is a submultigroup of the intersection between the two submultigroups. However, the normality and commutativity of commutator submultigroups were not studied in the paper. In this paper, among other related results, we show that commutator submultigroups is commutative and normal.

2. Preliminaries

Definition 2.1 ([3]) Let X be a set. A multiset G over X is a cardinal-valued function, $C_G : X \longrightarrow N = \{0, 1, \dots\}$ such that for $x \in Dom(G)$, $\Rightarrow G(x)$ is cardinal and $G(x) = C_G(x) > 0$, where $C_G(x)$ denotes the number of times an object x occur in multiset G, that is a count function of G (where $C_G(x) = 0$, implies $x \notin Dom(G)$). The set X is called the ground or generic set of the class of all multisets containing objects from X. All set of multisets over X is denoted as $MS(X)$.

Definition 2.2 ([6]) Let G and H be two multisets over X. Then H is called a submultiset of G written as $H \subseteq G$ if $C_H(x) \leq C_G(x), \forall\, x \in X$. Also, if $H \subseteq G$ and $H \neq G$, then H is called a proper submultiset of G denoted as $H \subset G$. A multiset is called the parent in relation to its submultiset.

Definition 2.3 ([2]) If G and H be two multisets over X, then the intersection and union of H and G, denoted by $H \cap G$ and $H \cup G$, respectively are defined by the rules that for any object $x \in X$;

i. $C_{H\cap G}(x) = C_H(x) \wedge C_G(x)$,

ii. $C_{H\cup G}(x) = C_H(x) \vee C_G(x)$.

where $\wedge$ and $\vee$ represent minimum and maximum respectively.

Definition 2.4 ([12]) Let X be a group. A multiset G over X is said to be a multigroup over X if the count function G or C_G satisfies the following conditions;

i. $C_G(xy) \geq C_G(x) \wedge C_G(y),\ \forall x, y \in X,$

ii. $C_G(x^{-1}) \geq C_G(x),\ \forall x \in X.$

Equivalently, $C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y), \forall\, x, y \in X$ the set of all multigroups over X is denoted by $MG(X)$.

Definition 2.5 ([12]) Given that X is a group and $G \in MG(X)$, then the root of $G$is

$$G_* \coloneqq \{x \in X : \ C_G(x) \geq 0\}$$

Definition 2.6 ([10]) Let $G \in MG(X)$. A submultiset H of G is called a submultigroup of G denoted by $H \sqsubseteq G$, if H is a multigroup. A submultigroup H of G is a proper submultigroup denoted by $H \sqsubset G$, if $H \sqsubseteq G$ and $G \neq H$.

Definition 2.7 ([4]) A submultigroup H of G is said to be a normal submultigroup of G, if and only if $h \in H_* : C_H(xhx^{-1}) \geq C_H(h), \forall x \in G_*$.

Definition 2.8 ([4]) Let $\{A_i\}_{i\in I}, I = 1, 2, \dots, n$ be an arbitrary family of multigroups of X. Then $C_{\cap_{i\in I}} A_i(x) = \bigwedge_{i\,\in\, I} C_{A_i}(x),\ x \in X$ and $C_{\cup_{i\in I}} A_i(x) = \bigvee_{i\,\in\, I} C_{A_i}(x),\ x \in X.$

The family of multigroups $\{A_i\}_{i\in I}$ of X is said to have inf/sup assuming chain if either $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$ respectively.

Proposition 2.1 ([11]) Let H be a submultigroup of $G \in MG(X)$, then the following statements are equivalent;

i. H is a normal submultigroup of G,

ii. $C_H(xhx^{-1}) \geq C_H(h), \forall\, x, y \in X,$

iii. $C_G(xy) = C_G(yx), \forall\, x, \in X.$

Definition 2.9 ([12]) Let $G \in MG(X)$. Then G is said to be abelian if

$$C_G(xy) = C_G(yx), \forall\, x, y \in X.$$

Remark Every normal submultigroup is commutative but the converse is not true.

Example 2.1 Let $X = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ be a group under matrix multiplication for

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad g_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\quad g_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad g_6 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\quad g_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad g_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

$A = [g_1{}^{10}, g_2{}^5, g_3{}^7, g_4{}^5, g_5{}^5, g_6{}^5, g_7{}^7, g_8{}^8] \in MG(X)$ and

$N = [g_1{}^7,\ g_3{}^3,\ g_7{}^3,\ g_8{}^4] \leq A$. Hence,

$C_N(n_1n_2) = C_N(n_2n_1),\ n_1, n_2 \in N_*$ but

$\exists n \in N_* : \ C_N(nxn^{-1}) < C_N(x), \qquad x \in X.$

Definition 2.10 ([12]) Let $A, B \in MG(X)$. Then the product of A and B denoted by $A \circ B$ is defined by $C_{A\circ B}(x) = \vee\{C_G(y) \wedge C_G(z) : x = yz, y, z \in X\}$.

Definition 2.11 ([9]) Let X and Y be two groups and let $\theta : X \longrightarrow Y$ be a homomorphism. Suppose A and B are multigroups of X and Y respectively. Then θ induces a homomorphism from A to B which satisfies;

i. $C_{\theta(A)}(y_1y_2) \geq C_{\theta(A)}(y_1) \wedge C_{\theta(A)}(y_2), \forall\, y_1, y_2 \in Y,$

ii. $C_B\big(\theta(x_1x_2)\big) \geq C_B\big(\theta(x_1)\big) \wedge C_B\big(\theta(x_2)\big), \forall\, x_1, x_2 \in X$, where

(i) the image of A under θ, denoted by $\theta(A)$, is a multiset of Y defined by

$$C_{\theta(A)}(y) = \begin{cases} \vee_{x\in\theta^{-1}(y)} C_A(x), & \theta^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise for each } y \in Y \end{cases}$$

(ii) the inverse image of B under θ, denoted by $\theta^{-1}(B)$, is a multiset of X defined by $C_{\theta^{-1}(B)}(x) = \ C_B\big(\theta(x)\big), \forall\, x \in X.$

Definition 2.12 ([10]) Let $G \in MG(X)$ and $A, B \sqsubseteq G$. Then the commutator of A and B is the multiset (A, B) of X defined as follows:

$$C_{(A,B)}(x) = \begin{cases} \bigvee \{C_A(a) \wedge C_B(b) : x = [a,b]\} \\ 0, \qquad\qquad \text{otherwise} \end{cases}$$

Equivalently, $C_{(A,B)}(x) = \vee[C_A(a) \wedge C_B(b) : x = aba^{-1}b^{-1}]$

Since the supremum of an empty set is zero, $C_{(A,B)}(x) = 0$ if x is not a commutator.

Remark 2.2 Let $G \in MG(X)$ and $A, B \sqsubseteq G$. Then the commutator of A and B is a multigroup generated by $C_{(A,B)}(x)$.

Theorem 1 ([10]) Let A and B be normal submultigroups of $G \in MG(X)$. Then $[A,B] \sqsubseteq A \cap B$.

3. Main Results

Definition 3.1 Let $\boldsymbol{G} \in MG(X)$. The commutator of G denoted by $\boldsymbol{\gamma_{[G]}}$, is a submultigroup

$$C_{\gamma_{[G]}}(x) := \begin{cases} \vee\{\{C_G(a) \wedge C_G(b), \text{if } x = [a,b]\} \\ \quad 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Theorem 3.1 Let $G \in MG(X)$, then $C_{\gamma_{[G]}}(xy) = C_{\gamma_{[G]}}(yx)$ if $\forall x, y \in (\gamma_{[G]})_*$

Proof. Let $G \in MG(X)$ and $\gamma_{[G]} \sqsubseteq G$. Let $\forall z \in (\gamma_{[G]})_*$ such that

$$z = xyx^{-1}y^{-1}, x, y \in (\gamma_{[G]})_* \Longrightarrow xy = yx$$

$$C_{\gamma_{[G]}}((xyx^{-1})x) = C_{\gamma_{[G]}}(xy)$$

Thus $C_{\gamma_{[G]}}(x^{-1}(xyx)) = C_{\gamma_{[G]}}(yx)$.

Hence $C_{\gamma_{[G]}}(xy) = C_{\gamma_{[G]}}(yx)$ it shows that $\gamma_{[G]}$ is a commutative submultigroup

Theorem 3.2 Suppose $G \in MG(X)$, then $\gamma_{[G]}$ is a normal submultigroups.

Proof. Given that $G \in MG(X)$.

If $x \in \{[a,b] : a, b \in X\}$, then $C_{\gamma_{[G]}}(x) = \vee\{\{C_G(a) \wedge C_G(b)\}\}$.

$C_{\gamma_{[G]}}(ab) = C_{\gamma_{[G]}}(ba), \forall a, b \in X$.

Now, let $x \in X$ and $c \in (\gamma_{[G]})_*$ then $C_{\gamma_{[G]}}(xcx^{-1}) = C_{\gamma_{[G]}}(xc)x^{-1}$

$$\geq C_{\gamma_{[G]}}(cx) \wedge C_{\gamma_{[G]}}(x^{-1}) = C_{\gamma_{[G]}}(c)$$

Thus $C_{\gamma_{[G]}}(xcx^{-1}) = C_{\gamma_{[G]}}(c^{-1}), \forall x \in X$.

Hence $\gamma_{[G]}$ is a normal submultigroup of G.

Theorem 3.3 The commutator submultigroup of every multigroup G over a group X is trivial if X is abelian.

Proof. Let $G \in MG(X)$. For any $x \in (\gamma_{[G]})_*$, we have

$$\begin{aligned} C_{\gamma_{[G]}}(x) &= \vee\{C_G(a) \wedge C_G(b) : a, b \in X\} \\ &= C_{\gamma_{[G]}}(aba^{-1}b^{-1}) \\ &= C_{\gamma_{[G]}}(baa^{-1}b^{-1}) = C_{\gamma_{[G]}}(beb^{-1}) \\ &= C_{\gamma_{[G]}}(bb^{-1}) = C_{\gamma_{[G]}}(e) \end{aligned}$$

$\Longrightarrow xy = yx \in X$. Hence $\forall x, y \in X, C_G(xy) = C_G(yx)$.

Theorem 3.4 Suppose $G \in MG(X)$ and $A, B \sqsubseteq G$. If $A \sqsubseteq B$ then $[A,C] = [B,C]$ for every $C \sqsubseteq G$. For $[A,C]$: $C_{(A,C)}(x) = \vee[C_A(a) \wedge C_C(b) : x = aca^{-1}c^{-1}]$

Proof. Let $A, B \sqsubseteq G \in MG(X)$ such that $C_A(x) \leq C_B(x), \forall x \in X$. For any submultigroup C of G. If $x \notin \{[a,b], a, b \in X\}$, then we have $C_{[A,C]}(x) = 0 = C_{[B,C]}(x)$.

Otherwise, let $x \in \{[a,b], a, b \in X\}$, then we have

$$\begin{aligned} C_{(A,C)}(x) &= \vee\{\{C_A(a) \wedge C_C(c)\} : x = [a,c]\} \\ &= \vee\{\{C_B(b) \wedge C_C(c)\} : x^{-1} = [b,c]\} = C_{(B,C)}(x). \end{aligned}$$

Hence $C_{(A,C)}(x) = C_{(B,C)}(x)$ if $x \in \{[a,b]: a, b \in X\}$.

Therefore, $[A,C] = [B,C]$.

Theorem 3.5 Let $G \in MG(X)$ such that A and C are submultigroups of G. Suppose B is any submultigroup of G, then $[A \circ B, C] = [A, C] \circ [B, C]$.

Proof. For $G \in MG(X)$ and A, C be submultigroups of G. Let $x = \{[a, b], a, b \in X\}$. Then

$$\begin{aligned} C_{(A\circ B,C)}(x) &= \vee \{\{C_{(A\circ B)}(a) \wedge C_C(b)\} : x = [a, b]\} \\ &= \vee \{\vee \{\{C_A(u) \wedge C_B(v) : a = uv\} \wedge C_C(b)\} : x = [a, b]\} \\ &= \vee \{\vee \{\{C_A(u) \wedge C_C(b)\} \wedge \{C_B(v) \wedge C_C(b)\}\}\} \\ &= \vee \{\{C_A(u) \wedge C_C(b)\} \wedge \{C_B(v) \wedge C_C(b)\}\} \\ &= \vee \{\{C_{(A,C)}[u, b] \wedge C_{(B,C)}[v, b]\}: x = [uv, b]\} \end{aligned}$$

By Theorem 3.1 for any $x \in X, C_{(A,C)}(xyx^{-1}) \geq C_{(A,C)}(y), y \in ([A, C])_*$.

$$\begin{aligned} &= \vee \{\{C_{(A,C)}(y) \wedge C_{(B,C)}(z)\}, x = yz\} \\ &= C_{(A,C)\circ(B,C)}(x). \end{aligned}$$

Now, since $[A, C]$ is normal submultigroup of G then $[A, C] \circ [B, C] \sqsubseteq G$.
Hence, $C_{(A\circ B,C)}(x) = C_{(A,C)\circ(B,C)}(x), \forall x \in \{[a, b], a, b \in X\}$.

Theorem 3.6 Let $G \in MG(X)$ and $A, B \sqsubseteq G$, then $([A, B])_* = [A_*, B_*]$.

Proof Suppose $A, B \sqsubseteq G \in MG(X)$, then there exists $a, b \in X$ such that $C_{(A,B)}(x) > 0$ for any $x = [a, b]\ and C_A(a) \wedge C_B(b) > 0$.
Consequently, $a \in A_*, b \in B_*$ and $x \in \{[l, m]: l \in A_*, m \in B_*\}$. Conversely, if $l \in A_*, m \in B_*$, then $0 < C_A(l) \wedge C_B(m) \leq C_{[A,B]}([l, m])$

$$\Longrightarrow C_{[A,B]}(x) > 0, \forall\, x = [l, m].$$

Hence, $\{[l, m]: l \in A_*, m \in B_*\} = ([A, B])_*$.
Since, $([A, B])_* \leq X$ and $[A_*, B_*]$ is the least subgroup of X containing $\{[l, m]: l \in A_*, m \in B_*\}$
Then we have $[A_*, B_*] \leq ([A, B])_*$. On the other hand,

$$\{[l, m]: l \in A_*, m \in B_*\} \leq [A_*, B_*].$$

If $\{[l, m]: l \in A_*, m \in B_*\} < [A_*, B_*]$ and there exists $x \notin \{[l, m]: l \in A_*, m \in B_*\}$ then

$$C_A(a) \wedge C_B(b) < 0, x = [a, b] \Longrightarrow \{[l, m]: l \in A_*, m \in B_*\} \neq ([A, B])_*.$$

Conversely, let $\{[l, m]: l \in A_*, m \in B_*\} = [A_*, B_*]$ then $C_{[A,B]}(x) > 0, \forall\, x = [l, m]$.
Since $([A, B])_* = \{[l, m]: l \in A_*, m \in B_*\} \leq [A_*, B_*]$.
Therefore, $([A, B])_* = [A_*, B_*]$.

Theorem 3.7 Given that $G \in MG(X)$ and $A, B \sqsubseteq G$, then $[A, B] = [B, A]$.

Proof Let $A, B \sqsubseteq G \in MG(X)$. If $x \notin \{[a, b], a, b \in X\}$, then
$C_{(A,B)}(x) = 0 = C_{(B,A)}(x^{-1})$. Now, let $x \in \{[a, b], a, b \in X\}$, then

$$\begin{aligned} C_{(A,B)}(x) &= \vee \{\{C_A(a) \wedge C_B(b)\}, x \in \{[a, b], a, b \in X\}\} \\ &= \vee \{\{C_B(b) \wedge C_A(a)\}, x^{-1} = [b, a]\} = C_{[B,A]}(x^{-1}). \end{aligned}$$

Since $C_G(y^{-1}) = C_G(y)$, for all $y \in X$. This imply that
$C_{(B,A)}(x^{-1}) = C_{(B,A)}(x), \forall x \in \{[a, b] : a, b \in X\}$
Hence $C_{(A,B)}(x) = C_{(B,A)}(x)$ this shows that $[A, B] = [B, A]$.

Theorem 3.8 Suppose $G \in MG(X)$ and $A, B \sqsubseteq G$. Then $[A, B] \subseteq A \cup B$.

Proof Let $A, B \sqsubseteq G \in MG(X)$, Let $x \in X$. Suppose $x \notin \{[a, b], a, b \in X\}$ then

$$C_{(A,B)}(x) = 0 \leq C_A(a) \vee C_B(b).$$

Otherwise, $C_{(A,B)}(x) \geq 0$. For $x = [a, b]$ for some $a, b \in X$ then we have

$$\begin{aligned} C_A(x) &= C_A(aba^{-1}b^{-1}) \\ &\geq C_A(a) \wedge C_A(ba^{-1}b^{-1}) \end{aligned}$$

Since for any $y \in X$, $C_{(A,B)}(yxy^{-1}) = C_{(A,B)}(x), x \in ([A, B])_*$ then

$$= C_A(a) \wedge C_A(a^{-1}).$$

Hence $C_A(x) \geq C_A(a)$. Similarly, $C_B(x) \geq C_B(a)$.
Consequently, we have $C_A(a) \vee C_B(b) \leq C_A(x) \vee C_B(x) = C_{A\cap B}(x)$.
Therefore, $C_{[A,B]}(x) \leq C_{A\cup B}(x)$, $\forall x \in \{[a, b], a, b \in X\}$. Hence $[A, B] \subseteq A \cup B$.

Theorem 3.9 Suppose $G \in MG(X)$ and for any $A, B \sqsubseteq G$. Then $[A, B] \sqsubseteq A \cap B$
Proof Let $A, B \sqsubseteq G \in MG(X)$, show that $C_{(A,B)}(x) \leq C_{A\cap B}(x), x \in \{[a,b], a, b \in X\}$.
For any $A, B \sqsubseteq G$, we have for any $y, z \in X$, $C_{A\cap B}(yz^{-1}) \geq C_{A\cap B}(y) \wedge C_{A\cap B}(z)$.
Let $x \in X$ such that $x \notin \{[a,b], a, b \in X\}$ then $C_{(A,B)}(x) = 0 \leq C_A(a) \wedge C_B(b)$.
If $x = [a, b]$ for some $a, b \in X$ then $C_A(x) = C_A(aba^{-1}b^{-1})$

$$\geq C_A(a) \wedge C_A(ba^{-1}b^{-1})$$

Since for any $y \in X$, $C_{(A,B)}(yxy^{-1}) = C_{(A,B)}(x), x \in ([A,B])_* = C_A(a) \wedge C_A(a)$.
Hence, $C_A(x) \geq C_A(a)$. Similarly, $C_B(x) \geq C_B(a)$.
Consequently, $C_A(a) \wedge C_B(b) \leq C_A(x) \wedge C_B(x) = C_{A\cap B}(x), x \in X$
Therefore, $C_{(A,B)}(x) \leq C_{A\cap B}(x)$, $\forall x \in \{[a,b], a, b \in X\}$.

Theorem 3.10 Suppose $G \in MG(X)$ and $A, B \sqsubseteq G$ then $[A, B] \sqsubseteq A \circ B$.
Proof Let $A, B \sqsubseteq G \in MG(X)$ and $x \in X$. If $x \notin \{[a,b], a, b \in X\}$ then

$$C_{(A,B)}(x) = 0 \leq C_{A\circ B}(x).$$

Otherwise, $C_{(A,B)}(x) \geq 0$. Suppose $x = [a, b]$ for some $a, b \in X$ then we have

$$C_A(x) = C_A(aba^{-1}b^{-1}) \geq C_A(a) \wedge C_A(ba^{-1}b^{-1})$$

By Theorem 3.2 $[A, B]$ is a normal submultigroup, Hence, $C_A(x) = C_A(a^{-1}) \wedge C_A(a)$.
Thus $C_A(x) \geq C_A(a)$ and also, $C_B(x) \geq C_B(a)$ consequently,

$$C_{A\circ B}(x) = \vee\{C_A(c) \wedge C_B(d) : x = cd\}$$
$$\geq \vee\{\{C_A(x) \wedge C_B(x)\} : x = [a,b]\}$$
$$= C_{A\circ B}(x).$$

Hence, $C_{A\circ B}(x) \geq C_{A\circ B}(x), \forall\, x \in \{[a,b], a, b \in X\}$.

Theorem 3.11 For any $G \in MG(X)$ such that $A, B \sqsubseteq G$. If α and β are the maximum count in A and B respectively, then the maximum count of $[A, B]$ is the minimum count between α and β.
Proof Suppose $G \in MG(X)$ and τ be the maximum count of $[A, B]$. Given that $x \notin \{[a,b] : a, b \in X\}$, then $C_{(A,B)}(x) = 0 \leq \tau$. Now, suppose $x \in \{[a,b] : a, b \in X\}$, then

$$C_{(A,B)}(x) = \vee\{\{C_A(a) \wedge C_B(b)\} : x = [a,b]\}$$
$$\leq \{\vee\{C_A(a) : a \in A_*\} \wedge \{\vee\{C_B(b)\} : b \in B_*\}\}$$
$$= \vee\{C_{(A,B)}(x), \forall x \in \{[a,b] : a \in A_*, b \in B_*\} = \tau.$$

Hence, $C_{(A,B)}(x) \leq \tau$.
Let σ be the maximum count of $[A, B]$ such that $\sigma \leq \tau$. If $\sigma < \tau$ then we have $\sigma < \alpha$ and $\sigma < \beta$. Then there exists $a, b \in X$ such that $\sigma < C_A(a)$, $\sigma < C_B(b)$.
Consequently, $\sigma < \{C_A(a) \wedge C_B(b)\} \leq C_{(A,B)}([a,b])$, this contradicts that $\sigma \neq \tau$.
Hence, $\sigma = \tau$.

Theorem 3.12 Let X and Y be groups and $\theta : X \longrightarrow Y$ be a homomorphism. If $G \in MG(X)$ such that for any $A, B \sqsubseteq G$, then $[\theta(A), \theta(B)] = \theta([A, B])$.
Proof Given that $y \in Y$ and $\theta^{-1}(y) = \emptyset$, then $C_{\theta([A,B])}(y) = 0 \leq C_{[\theta(A),\theta(B)]}(y)$.
So, let $y = \theta(x)$ for some $x \in X$, then we have

$$C_{(A,B)}(x) = \vee\{\{C_A(a) \wedge C_B(b)\} : x = [a,b]\}$$
$$\leq \vee\{\{C_{\theta(A)}(\theta(a)) \wedge C_{\theta(B)}\theta((b))\} : y = [\theta(a), \theta(b)]\}$$
$$= \vee\{\{C_{\theta(A)}(c) \wedge C_{\theta(B)}(d)\} : y = [c,d]\}$$
$$= C_{(\theta(A),\theta(B))}(y), \ \forall\, y \in \{[c,d] : c, d \in Y\}$$

Therefore, $C_{\theta((A,B))}(y) = \vee\{C_{(A,B)}(x) : y = \theta(x)\} \leq C_{[\theta(A),\theta(B)]}(y)$.
Thus, $\theta([A, B]) \subseteq [\theta(A), \theta(B)]$.
Conversely, let $y \notin \{[c,d] : c, d \in Y\}$, then $C_{(\theta(A),\theta(B))}(y) = 0 \leq C_{\theta((A,B))}(y)$. So, for $y = [c, d]$. If either $\theta^{-1}(c) = \emptyset$ or $\theta^{-1}(d) = \emptyset$, then $C_{\theta(A)}(c) \wedge C_{\theta(B)}(d) = 0$.
Otherwise, it result to

$$C_{\theta(A)}(c) \wedge C_{\theta(B)}(d) = (\vee\{C_A(\theta^{-1}(c)) : c = \theta(a)\}) \wedge (\{\vee\{C_B(\theta^{-1}(d)) : d = \theta(b)\})$$

$$= \vee \{\{C_A(a) \wedge C_B(b)\} : c = \theta(a), d = \theta(b)\}$$
$$\leq \vee \{C_{(A,B)}([a,b]) : y = \theta([a,b])\}$$
$$\leq \vee \{C_{(A,B)}(x) : x = [a,b]\}$$
$$= C_{\theta([A,B])}(y).$$

Thus, $[\theta(A), \theta(B)] \subseteq \theta([A,B])$ and hence $[\theta(A), \theta(B)] = \theta([A,B])$

Theorem 3.13 Let X and Y be groups and $\theta : X \longrightarrow Y$ be a homomorphism. If $G \in MG(X)$ such that for any $A, B \sqsubseteq G$, $[\theta^{-1}(A), \theta^{-1}(B)] \subseteq \theta^{-1}([A,B])$.

Proof Given that $x \in X$ and $x \notin \{[a,b] : a, b \in X\}$, then $C_{(\theta^{-1}(A),\theta^{-1}(B))}(x) = 0 \leq C_{\theta^{-1}((A,B))}(x)$. Otherwise,

$$C_{(\theta^{-1}(A),\theta^{-1}(B))}(x) = \vee \left\{\{C_A(\theta(a)) \wedge C_B(\theta(b))\} : x = [a,b]\right\}$$
$$= \vee \left\{\{C_A(\theta(a)) \wedge C_B(\theta(b))\} : \theta(x) = [\theta(a), \theta(b)]\right\}$$
$$\leq \vee \{\{C_A(c), C_B(d)\} : \theta(x) = [c,d]\}$$
$$\leq C_{[A,B]}(\theta(x))$$
$$= C_{\theta^{-1}([A,B])}(x).$$

Therefore, $C_{[\theta^{-1}(A),\theta^{-1}(B)]}(x) \leq C_{\theta^{-1}([A,B])}(x)$. Since

$$C_{\theta^{-1}([A,B])}(cd) \geq C_{\theta^{-1}([A,B])}(c) \wedge C_{\theta^{-1}([A,B])}(d), \forall\, c, d \in Y.$$

Hence, $[\theta^{-1}(A), \theta^{-1}(B)] \subseteq \theta^{-1}([A,B])$.

4. Conclusion

In this paper, we have presented that the commutator submultigroup is commutative and normal. We also show that the commutator of the homomorphic image equals the image of the commutator. Therefore the concept of commutator submultigroup can be explore in studying multigroups that is defined on series as related to classical group

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